Existence of the Chapman-Enskog solution and its relation with first-order dissipative fluid theories

A. L. García-Perciante[1], A. R. Méndez[1], and O. Sarbach[1,2]
[1]*Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana-Cuajimalpa (05348) Cuajimalpa de Morelos, Ciudad de México México. and*
[2]*Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, Ciudad Universitaria, 58040 Morelia, Michoacán, México.*
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The conditions for the existence of the Chapman-Enskog first-order solution to the Boltzmann equation for a dilute gas are examined from two points of view. The traditional procedure is contrasted with a somehow more formal approach based on the properties of the linearized collision operator. It is shown that both methods lead to the same integral equation in the non-relativistic scenario. Meanwhile, for relativistic systems, the source term in the integral equation adopts two different forms. However, as we explain, this does not lead to an inconsistency. In fact, the constitutive equations that are obtained from both methods are shown to be equivalent within relativistic first-order theories. The importance of stating invariant definitions for the transport coefficients in this context is emphasized.

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I. INTRODUCTION AND PRELIMINARIES

The study of perturbative solutions to the Boltzmann equation in the dilute gas scenario dates back to work by Hilbert, specifically in relation to the so called 6th problem [1, 2]. In particular, the Hilbert expansion, suitably modified by Chapman and Enskog, led to one of the most successful methods of solution for the Boltzmann equation in dilute systems: the Chapman-Enskog (CE) approximation [3, 4]. Despite criticisms on its mathematical weaknesses, mostly on regimes beyond Navier-Stokes (see for example [2, 4, 5]), this method has been shown to be a powerful tool capable of fundamenting the hydrodynamic equations and delivering constitutive relations for dissipative fluxes which agree with experimental results in the linear regime.

In the relativistic scenario, the CE approximation was applied by pioneers in the field such as W. Israel [6] and S. R. de Groot [7]. However, the fact that its traditional form predicts first-order couplings between dissipative fluxes and spatial gradients of the state variables leads to its partial dismissal, based on the findings that such relations give rise to pathological theories for which the equilibrium configurations are generically unstable [8]. Recently, a new family of first-order theories has been proposed and shown to contain promising candidates to describe high temperature gases in curved spacetimes. These new theories allow one to consider a general frame and representation in which constitutive equations feature all possible couplings, including relations which involve time derivatives of the state variables. Under suitable restrictions, these theories have been shown to lead to physically sound equations [9, 10]. However, despite recent attempts [11, 12], their microscopical foundations are still not well understood from the point of view of the CE expansion. The purpose of this work is to shed new light on this problem.

As a starting point we consider the Boltzmann equation, which is an integrodifferential relation describing the balance between the time evolution of the one-particle distribution function f and the cumulative effects of collisions, and which we write here in a general form as

$$L_F\left[f\right] = Q\left(f f'\right). \tag{1}$$

Here L_F represents the Liouville operator, giving the total time derivative of the distribution function in the presence of an external electromagnetic force (indicated symbolically by F), and $Q\left(ff'\right)$ is an integral operator accounting for the balance of particles in a cell of phase space due to binary interactions, see Refs. [3, 13] for details. The CE expansion considers small perturbations of a local equilibrium state $f^{(0)}$ such that $f = f^{(0)}\left(1+\phi\right)$. Up to first order, $f^{(0)}$ and ϕ satisfy

$$Q\left(f^{(0)} f^{(0)\prime}\right) = 0, \tag{2}$$

and

$$C\left(\phi\right) = L_F\left[\ln f^{(0)}\right], \tag{3}$$

where $C\left(\phi\right)$ is the linearization of the operator $Q\left(ff'\right)$ at $f^{(0)}$ (see below for more details). Notice that we assume in this article that the scale of the electromagnetic field is macroscopic. In this work, we focus on the source term $L_F[\ln f^{(0)}]$ which needs to lie in the image of the operator C. We achieve this using two different methods.

Firstly, the traditional procedure is implemented, which performs an expansion of the convective time derivative term in the right-hand side of Eq. (3) [3, 7, 13]. The coefficients of this expansion are adjusted by integrating the first-order equation, arXiv:2412.03712v1 [gr-qc] 4 Dec 2024

Relativistic dissipative fluids in the trace-fixed particle frame: hyperbolicity, causality and stability

J. Félix Salazar,[1] Ana Laura García-Perciante,[1] and Olivier Sarbach[1,2]

[1] *Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana-Cuajimalpa (05348) Cuajimalpa de Morelos, Ciudad de México México.*

[2] *Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, Ciudad Universitaria, 58040 Morelia, Michoacán, México.*

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We propose a first-order theory of dissipative fluids in the trace-fixed particle frame, which is similar to Eckart's frame except that the temperature is determined by fixing the trace of the stress-energy tensor. Our theory is hyperbolic and causal provided a single inequality holds. For low wave numbers, the expected damped modes in the shear, acoustic, and heat diffusion channels are recovered. Stability of global equilibria with respect to all wave numbers is also analyzed. The conditions for hyperbolicity, causality and stability are satisfied for a simple gas of hard spheres or disks.

I. INTRODUCTION

Relativistic dissipative hydrodynamics plays a prominent role in many current problems in physics, including the description of cosmological fluids in the Universe [1, 2], the modeling of accretion disks [3–5] and the study of quark-gluon plasmas encountered under extreme laboratory conditions [6]. In order to address these problems, one requires a theory which is physically sound, i.e. one that is described by hyperbolic evolution equations with causal propagation and for which equilibrium states are stable. There are many different proposals [7–13] and a vast literature on this subject, see for instance [14–17] for reviews. In particular, physically sound theories which are second-order in the off-equilibrium quantities have been developed in [8, 9, 18]. Recently, there has been a vivid interest in theories, referred to as BDNK [19–26], which are first-order in the gradient expansion of the state variables. Instead of taking the entropy principle as a starting point [8, 27, 28], BDNK assume general couplings between the non-equilibrium quantities and the gradients of the state variables. Provided the coupling constants satisfy an ample set of nontrivial inequalities, a physically sound theory in the aforementioned sense is obtained.

In this article we present a formalism similar to BDNK. However, in contrast to this theory, our approach is based on the use of a specific frame, namely the trace-fixed particle frame, in which the state variables (n, T, u^μ) (particle number density, temperature parameter and mean-particle velocity) are fixed using the current density vector and the trace of the stress-energy tensor. This particular choice can be motivated by means of a kinetic (microscopic) formalism, whose details will be published elsewhere [29], and it presents several advantages as we explain in the following.

Our theory, which describes a simple non-degenerate dissipative fluid propagating on a curved spacetime and electromagnetic background, is hyperbolic and causal, provided the *single* inequality (30) is satisfied. Furthermore, we establish general conditions which guarantee that, for large enough values of the only free parameter Λ_0, global equilibrium states in flat spacetime are stable with respect to modes of arbitrary wave numbers. Additionally, the known propagation of modes in the shear, acoustic and heat channels with low wave numbers is recovered independently of the value of Λ_0. We verify the fulfillment of the fundamental inequality (30) and the stability conditions for all temperatures in the case of a simple gas of hard spheres or disks in three and two dimensions, respectively.

We work on a fixed, globally hyperbolic and time-oriented $(d+1)$-dimensional spacetime with $d \geq 2$. Greek indices $\mu, \nu, \ldots$ run over $0, 1, \ldots, d$ and ∇ denotes the Levi-Civita connection associated with the spacetime metric $g_{\mu\nu}$. $F^{\mu\nu}$ refers to the background electromagnetic field and q to the charge of the fluid constituents. We use geometrized units and the signature convention $(-, +, \ldots, +)$ for the metric.

II. FLUID EQUATIONS

The equations of motions for a relativistic charged fluid are given by

$$\nabla_\mu J^\mu = 0, \qquad \nabla_\mu T^{\mu\nu} + q J_\mu F^{\mu\nu} = 0, \tag{1}$$

and in this article we work with a current density and stress-energy tensor which have the form

$$J^\mu = n u^\mu, \tag{2}$$

$$T_{\mu\nu} = (ne + \epsilon)\, u_\mu u_\nu + \left(p + \frac{\epsilon}{d}\right) \Delta_{\mu\nu} + 2 u_{(\mu} \mathcal{Q}_{\nu)} + \mathcal{T}_{\mu\nu}. \tag{3}$$

Here, $\Delta_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu$, $u^\mu u_\mu = -1$, and $()$ denotes symmetrization. The internal energy density per particle e is assumed to be a function of T only, and the pressure is determined through the ideal gas equation of state $p = n k_B T$, where k_B is the Boltzmann constant. Further, ϵ, $\mathcal{Q}^\mu$ (which is orthogonal to u^μ), and $\mathcal{T}_{\mu\nu}$ (which is symmetric, trace-free and orthogonal to u^μ) are off-equilibrium corrections. In particular, $\mathcal{Q}^\mu$ describes the heat flux and $\mathcal{T}_{\mu\nu}$ the trace-free part of the viscosity tensor. Meanwhile, the scalar non-equilibrium contribution ϵ accounts for the effects of the bulk viscosity, as we will see shortly.

The choice of frame (i.e. the assignment of n, T, and u^μ to a nonequilibrium state described by J^μ and $T_{\mu\nu}$) here adopted corresponds to (i) a particle frame, which fixes n and u^μ such

Relativistic dissipative fluids in the trace-fixed particle frame: strongly hyperbolic quasi-linear first-order evolution equations

J. Félix Salazar,[1] Ana Laura García-Perciante,[1] and Olivier Sarbach[1,2]

[1]*Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana-Cuajimalpa (05348) Cuajimalpa de Morelos, Ciudad de México México.*
[2]*Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, Ciudad Universitaria, 58040 Morelia, Michoacán, México.*

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In this paper we derive a new first-order theory of dissipative fluids by adopting the trace-fixed particle frame. Whereas in a companion paper we show that this theory is hyperbolic, causal and stable at global equilibrium states, here we prove that the full nonlinear system of equations can be cast into a first-order quasilinear system which is strongly hyperbolic. By rewriting the system in first-order form, auxiliary constraints are introduced. However, we show that these constraints propagate, and thus our theory leads to a well-posed Cauchy problem.

I. INTRODUCTION

A persistent challenge in theoretical physics is the formulation of a theory of dissipative fluids which aligns with both thermodynamics and (special or general) relativity. The need for obtaining such a formulation is reinforced by recent experiments and observations in high-energy physics and relativistic astrophysics. A prominent example is the understanding of the dynamics and transport properties of the quark-gluon plasma produced in heavy-ion colliders [1–3]. Moreover, the interpretation of data obtained from detections of gravitational radiation generated by binary neutron star [4, 5] or black hole-neutron star mergers [6], requires an accurate description of viscous fluids in the strong gravity regime [7, 8]. Also, it has been found that dissipative effects can significantly alter the properties of the outflow and electromagnetic signals in accretion processes [7].

A relativistic theory of dissipative fluids which is suitable for addressing the aforementioned problems should satisfy a minimal set of requirements. On the one hand, this entails that the evolution equations give rise to a well-posed Cauchy problem. In fact, this evolution system should be strongly hyperbolic to ensure finite speed of propagation, and also it needs to be causal, meaning that no physical modes propagate faster than the speed of light. On the other hand, mandatory requisites of the theory are the fulfillment of the second law of thermodynamics and the stability of equilibrium states, in the sense that small perturbations of global equilibrium configurations in Minkowski space decay in time. Satisfying all these criteria has proven to be a challenging task and has led to a plethora of proposals since the early approaches by Eckart and Landau-Lifshitz (see Refs. [9–12] for recent reviews).

Recently, a new formulation known in the literature as BDNK theory [13–20] has attracted a lot of attention. This theory is based on a gradient expansion in the state variables (n, ε, u^μ) (representing, respectively, the particle density, energy density, and four-velocity of the fluid when in equilibrium), or equivalent variables involving the chemical potential. When truncated to first order, and under suitable restrictions on the transport coefficients, the theory can be shown to satisfy the aforementioned requirements in the following sense: In [17], a strongly hyperbolic and causal system is obtained which is shown to give rise to nonnegative entropy production (within the limits of validity of the theory) and to stable global equilibria, provided the transport coefficients obey an ample set of inequalities. Also, Kovtun et. al. [15, 16, 18] obtain dissipative fluid theories which are causal, stable, and shown to be weakly hyperbolic. A key observation in BDNK theory is to relax the restriction of the Eckart or Landau-Lifshitz frames which are usually adopted and result in theories that are not only unstable but are not even hyperbolic [21]. The frame in [17] is partially determined by fixing n and u^μ to those matching the current density J^μ, i.e. such that $J^\mu = nu^\mu$; however, ε is not required to match the energy density measured by an observer comoving with u^μ, as in the Eckart frame. Instead, the stress-energy tensor is determined by imposing constitutive relations with parameters that must satisfy certain inequalities that enforce the desired properties of the system. Similarly, the work in [15, 16] does not impose specific matching conditions and determines the frame through suitable constitutive relations. For BDNK hydrodynamics arising from the fluid-gravity correspondence, see for instance Refs. [18, 22].

In this article, we adopt an approach which, although similar in spirit to BDNK theory, imposes matching conditions which completely fix the frame. As in [17], n and u^μ are fixed by the current density. However, in contrast to that work, we fix ε through the trace of the stress-energy tensor $T^{\mu\nu}$. More precisely, we work with state variables (n, T, u^μ) (with T representing the temperature when in equilibrium) which parametrize the equilibrium configuration that matches the expressions for J^μ and $T^\mu{}_\mu$. This choice, which we shall refer to as the *trace-fixed particle frame* (or TFP frame), can be motivated from kinetic theory by noticing that it fixes the first few moments of the distribution function [23]. Further, it is worthwhile to point out that this particular frame is well adapted to conformal fluids for which $T^\mu{}_\mu = 0$.

Besides the three transport coefficients η, ζ, and κ, denoting the shear and bulk viscosities and the thermal conductivity, respectively, our theory possesses two additional coefficients, Γ_1 and Γ_2. However, they do not affect the physical content of the theory up to and including terms which are first-order in the gradients of the state variables. In a companion article [24] we show that a suitable choice for Γ_2 leads to a system of

Gravitational atoms beyond the test field limit: The case of Sgr A* and ultralight dark matter

Miguel Alcubierre,[1] Juan Barranco,[2] Argelia Bernal,[2] Juan Carlos Degollado,[3] Alberto Diez-Tejedor,[2] Miguel Megevand,[4] Darío Núñez,[1,5] and Olivier Sarbach[6,7]

[1] *Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Circuito Exterior C.U., A.P. 70-543, Coyoacán, México 04510, CdMx, México*

[2] *Departamento de Física, División de Ciencias e Ingenierías, Campus León, Universidad de Guanajuato, C.P. 37150, León, México*

[3] *Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Apdo. Postal 48-3, 62251, Cuernavaca, Morelos, México*

[4] *Instituto de Física Enrique Gaviola, CONICET. Ciudad Universitaria, 5000 Córdoba, Argentina*

[5] *Departamento de Matemática da Universidade de Aveiro and Centre for Research and Development in Mathematics and Applications (CIDMA), Campus de Santiago, 3810-183 Aveiro, Portugal*

[6] *Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana-Cuajimalpa (05348) Cuajimalpa de Morelos, Ciudad de México, México*

[7] *Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, Ciudad Universitaria, 58040 Morelia, Michoacán, México*

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We construct *gravitational atoms* including self-gravity, obtaining solutions of the Einstein-Klein-Gordon equations for a scalar field surrounding a non-rotating black hole in a quasi-stationary approximation. We resolve the region near the horizon as well as the far field region. Our results are relevant in a wide range of masses, from ultralight to MeV scalar fields and for black holes ranging from primordial to supermassive. For instance, a system with a scalar field consistent with ultralight dark matter and a black hole mass comparable to that of Sagittarius A* can be modeled. A density *spike* near the event horizon, although present, is negligible, contrasting with the prediction in [P. Gondolo and J. Silk, *Phys. Rev. Lett.*, 83:1719–1722, 1999] for cold dark matter.

Introduction.—Efforts to disentangle dark matter (DM) properties using direct [1–3] or indirect [4–6] detection methods have reported null results, pointing towards the fact that DM may only interact through gravity. Experimental and observational searches have disfavored heavy DM candidates, including massive compact halo objects (MACHOs) and weakly interacting massive particles (WIMPs). This has led to growing interest in lighter DM candidates, often with masses below that of the proton. Examples of these include spin zero particles such as the QCD axion [7–10], for which $10^{-6}\,\mathrm{eV} < m_a c^2 < 10^{-3}\,\mathrm{eV}$, axion-like particles (ALPs) [11] with masses lighter than the QCD axion, $10^{-26}\,\mathrm{eV} < m_{\mathrm{ALP}} c^2 < 10^{-6}\,\mathrm{eV}$, and fuzzy DM [12], also known as ultralight DM or scalar field (SF) DM [13–15], where $m_{\mathrm{SFDM}} c^2 \sim 10^{-22}\,\mathrm{eV}$.

For scalar light candidates, a rich phenomenology arises when their reduced Compton wavelength $\lambda_\phi = \frac{\hbar}{m_\phi c}$ (m_ϕ being the DM particle mass) becomes macroscopic. In previous works [16–19], we have explored configurations of a SF surrounding a non-rotating black hole (BH), showing that, in the test field approximation they can form quasi-bound states provided that λ_ϕ is larger than twice the Schwarzschild radius $R_{\mathrm{Sch}} = \frac{2GM_{BH}}{c^2}$, i.e.

$$\left(\frac{4GM_{BH}}{c^2}\right)\left(\frac{m_\phi c}{\hbar}\right) = \frac{2R_{\mathrm{Sch}}}{\lambda_\phi} < 1\,, \tag{1}$$

where M_{BH} is the black hole mass. Furthermore, if the left-hand side of Eq. (1) is sufficiently small, this scalar cloud is not radiated away or rapidly swallowed by the BH; on the contrary, it can survive for cosmological times [17]. Moreover, its spectrum resembles that of the hydrogen atom, $(\hbar/m_\phi c^2)\omega_n = \sqrt{1-\alpha_G^2/n^2}$ [17, 20, 21], with α_G defined in Eq. (2) below. Configurations with these characteristics have been named scalar wigs in [17, 18], and are also known as gravitational atoms in the context of superradiance [22, 23].

By introducing a new set of coordinates that generalize the ingoing Eddington-Finkelstein ones to the self-gravitating case, in this work we further develop the approach of [19] and construct gravitational atoms in spherical symmetry including self-gravity, obtaining quasi-stationary solutions of the Einstein-Klein-Gordon (EKG) equations describing a SF that surrounds a BH.

In terms of the *gravitational fine structure constant* [24], defined by

$$\alpha_G \equiv \frac{GM_{BH}m_\phi}{\hbar c} = 7.5\times 10^9 \left(\frac{M_{BH}}{M_\odot}\right)\left(\frac{m_\phi c^2}{\mathrm{eV}}\right), \tag{2}$$

the condition (1) for the existence of SF clouds in the test field limit becomes $\alpha_G < 1/4$. Here, given BH and SF masses satisfying this inequality, we obtain a family of solutions parameterized by the SF amplitude A at the horizon. Although the solutions depend on these three parameters, configurations with the same α_G and A are related to each other by a simple re-scaling. This facilitates considering a wide set of astrophysical realizations (see Table I for relevant examples).

Nonrelativistic Proca stars: Spherical stationary and multi-frequency states

Emmanuel Chávez Nambo,[1] Alberto Diez-Tejedor,[2]
Edgar Preciado-Govea,[2] Armando A. Roque,[3,2] and Olivier Sarbach[1,4]

[1] *Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, Ciudad Universitaria, 58040 Morelia, Michoacán, México*
[2] *Departamento de Física, División de Ciencias e Ingenierías, Campus León, Universidad de Guanajuato, 37150, León, México*
[3] *Unidad Académica de Física, Universidad Autónoma de Zacatecas, 98060 Zacatecas, México*
[4] *Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana-Cuajimalpa, 05348 Cuajimalpa de Morelos, Ciudad de México, México*

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In this paper we follow an effective theory approach to study the nonrelativistic limit of a selfgravitating and selfinteracting massive vector field. Our effective theory is characterized by three parameters: the field's mass m_0 and the selfinteraction constants λ_n and λ_s. For definiteness, we focus on a systematic study of the equilibrium configurations, commonly referred to as Proca stars when they have finite energy. We identify two different types of Proca stars, depending on the specific sector of the effective theory that we are exploring. In the generic sector, defined by $\lambda_s \neq 0$, all equilibrium configurations are stationary states described by wave functions that evolve harmonically in time. However, in the symmetry-enhanced sector, for which $\lambda_s = 0$, there exist multi-frequency states whose wave functions oscillate with two or three distinct frequencies in addition to the stationary states. We determine the conditions under which a ground state configuration with fixed particle number exists. When these conditions are met, we prove that the lowest energy is reached by a stationary spherically symmetric configuration of constant polarization that is linear or circular depending on the sign of λ_s. We numerically construct some illustrative examples of spherical stationary and multi-frequency solutions, analyze their properties, and compare them with our analytical predictions. Unlike stationary states and other soliton configurations, which form a discrete set in the solution space associated with fixed particle number, the symmetry-enhanced sector exhibits a continuum of solutions with multi-frequency states connecting stationary states of constant polarization.

I. INTRODUCTION

Boson stars are regular, finite energy configurations that do not disperse in time and are encountered in massive, selfgravitating scalar field theories [1–8]. Furthermore, similar solutions arise in theories with higher rank fields, such as Proca stars in vector field theories. In this paper, we take an effective theory approach to investigate the nonrelativistic limit of a selfgravitating and selfinteracting massive vector field, focusing on equilibrium configurations and on spherically symmetric states in particular. The linear stability of these solutions will be analyzed in a follow-up paper [9].

Proca stars were first introduced by Brito, Cardoso, Herdeiro, and Radu in Ref. [10], where they constructed solutions of the Einstein-Proca equations with both static spherically symmetric and stationary axially symmetric spacetimes. This pioneering work triggered a surge in research on such stars that includes theoretical investigations [11–16], numerical simulations [17–19], and astrophysical applications [20–24]. As for this latter possibility, massive vector fields may be especially relevant to ultralight dark matter models [25–28], exhibiting a richer phenomenology compared to spin $s = 0$ axion-like particles [29]. In galaxies, these particles could form dark matter halos, whose global structure is inherently Newtonian, and this motivates our focus on the nonrelativistic theory in this paper.

In the nonrelativistic regime, Proca stars have been explored by Amin, Jain, and collaborators in [30–34] (see also Refs. [35–39]). We can think of these objects as selfgravitating condensates of spin $s = 1$ particles, where matter is described in terms of a vector-valued wave function $\vec{\psi}(t, \vec{x})$ satisfying the Schrödinger equation and gravity by the Newtonian gravitational potential $\mathcal{U}(t, \vec{x})$ obeying Poisson's equation. When selfinteractions are included, the Schrödinger equation needs to be replaced by a Gross-Pitaevskii type equation with two coupling constants λ_n and λ_s. We refer to these equations as the $s = 1$ *Schrödinger-Poisson system* when $\lambda_n = \lambda_s = 0$, and as the $s = 1$ *Gross-Pitaevskii-Poisson system* otherwise, and to the resulting finite energy equilibrium configurations as *nonrelativistic Proca stars*.

The spectrum of Proca star solutions depends on the spin-spin selfinteraction parameter λ_s. When $\lambda_s \neq 0$, which we henceforth call the *generic sector* of the theory, the Proca star's wave function evolves in time harmonically. As in standard quantum mechanics, we shall refer to these equilibrium configurations as *stationary* (or single-frequency) states. However, when $\lambda_s = 0$, the effective theory acquires an additional (accidental) symmetry, resulting in the *symmetry-enhanced sector*. In this sector, new types of equilibrium configurations appear besides the stationary states in which the wave function